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Geodesic Mappings and Differentiability of Metrics, Affine and Projective Connections

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Abstract. In this paper we study fundamental equations of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds with respect to the smoothness class of these geometric objects. We prove that the natural smoothness class of these problems is preserved.

1. Introduction and Basis Definitions

To theory of geodetic mappings and transformations were devoted many papers, these results are formulated in large number of researchs and monographs [1], [2], [3], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [16], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [30], [31], [32], [33], [34], [35], [36], [37], etc.

First we studied the general dependence of geodesic mappings of manifolds with affine and projective connection onto (pseudo-) Riemannian manifolds in dependence on the smoothness class of these geometric objects. We presented well known facts, which were proved by H. Weyl [37], T. Thomas [35], J. Mikeš and V. Berezovski [21], see [5], [20], [25], [26], [30], [32], [36].

In these results no details about the smoothness class of the metric, resp. connection, were stressed. They were formulated as "for sufficiently smooth" geometric objects.

In the paper [14, 15] we proved that these mappings preserve the smoothness class of metrics of geodetically equivalent (pseudo-) Riemannian manifolds. We prove that this property generalizes in a natural way for a more general case.

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2. Geodesic Mapping Theory for Manifolds with Affine and Projective Connections

Let $A_n = (M, \nabla)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla})$ be manifolds with affine connections ∇ and $\bar{\nabla}$, respectively, without torsion.

Definition 2.1. *A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a geodesic mapping of A_n onto $\bar{A}_n$ if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.*

A manifold A_n admits a geodesic mapping onto $\bar{A}_n$ if and only if the *Levi-Civita equations* (H. Weyl [37], see [5, p. 56], [25, p. 130], [26, p. 166], [32, p. 72]):

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a differential form on M ($= \bar{M}$). If $\psi \equiv 0$ then f is *affine* or *trivially geodesic*.

Eliminating ψ from the formula (1) T. Thomas [35], see [5, p. 98], [25, p. 132], obtained that equation (1) is equivalent to

$$\bar{\Pi}(X, Y) = \Pi(X, Y) \ \text{ for all tangent vectors } X, Y, \tag{2}$$

where

$$\Pi(X, Y) = \nabla(X, Y) - \frac{1}{n+1}\,(trace(V \to \nabla_V X) \cdot Y + trace(V \to \nabla_V Y) \cdot X)$$

is the *Thomas' projective parameter* or *Thomas' object of projective connection*.

A geometric object Π that transforms according to a similar transformation law as Thomas' projective parameters is called a *projective connection*, and manifolds on which an object of projective connection is defined is called a *manifold with projective connection*, denoted by P_n. Such manifolds represent an obvious generalization of affine connection manifolds.

A projective connection on P_n will be denoted by $\blacktriangledown$. Obviously, $\blacktriangledown$ is a mapping $TP_n \times TP_n \to TP_n$, i.e. $(X, Y) \mapsto \blacktriangledown_X Y$. Thus, we denote a manifold M with projective connection $\blacktriangledown$ by $P_n = (M, \blacktriangledown)$. See [5, p. 99], [6].

We restricted ourselves to the study of a coordinate neighborhood (U, x) of the points $p \in A_n$ (P_n) and $f(p) \in \bar{A}_n$ $(\bar{P}_n)$. The points p and $f(p)$ have the same coordinates $x = (x^1, \dots, x^n)$.

We assume that A_n, $\bar{A}_n$, P_n, $\bar{P}_n \in C^r$ $(\nabla, \bar{\nabla}, \blacktriangledown, \bar{\blacktriangledown} \in C^r)$ if their components $\Gamma^h_{ij}(x), \bar{\Gamma}^h_{ij}(x), \Pi^h_{ij}(x), \bar{\Pi}^h_{ij}(x) \in C^r$ on (U, x), respectively. Here C^r is the smoothness class. On the other hand, the manifold M which these structures exist, must have a class smoothness C^{r+2}. This means that the atlas on M is of class C^{r+2}, i.e. for the non disjunct charts (U, x) and (U', x') on $(U \cap U')$ it is true that the transformation $x' = x'(x) \in C^{r+2}$.

Formulae (1) and (2) in the common system (U, x) have the local form:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i \ \text{ and } \ \bar{\Pi}^h_{ij}(x) = \Pi^h_{ij}(x),$$

respectively, where ψ_i are components of ψ and δ^h_i is the Kronecker delta.

It is seen that in a manifold $A_n = (M, \nabla)$ with affine connections ∇ there exists a projective connection $\blacktriangledown$ (i.e. Thomas projective parameter) with the same smoothness. The opposite statement is not valid, for example if $\nabla \in C^r$ ($\Rightarrow \blacktriangledown \in C^r$ and also $\bar{\blacktriangledown} \in C^r$) and $\psi(x) \in C^0$, then $\bar{\nabla} \in C^0$.

In the paper [12] we presented a construction that the existing ∇ on M guarantees on $P_n = (M, \blacktriangledown)$. Moreover, the following theorem holds:

Theorem 2.2. *An arbitrary manifold $P_n = (M, \blacktriangledown) \in C^r$ admits a global geodesic mapping onto a manifold $\bar{A}_n = (M, \bar{\nabla}) \in C^r$ and, moreover, for which a formula* trace$(V \to \bar{\nabla}_V)X = \nabla_X G$ *holds for arbitrary X and a function G on M, i.e. $\bar{A}_n$ is an equiaffine manifold and $\bar{\nabla}$ is an equiaffine connection. Moreover, if $r \geq 1$ the Ricci tensor on $\bar{A}_n$ is symmetric.*

Proof. It is known that on the whole manifold $M \in C^{r+2}$ exists globally a sufficiently smooth metric $\hat{g} \in C^{r+1}$. For our purpose it is sufficient if $\hat{g} \in C^{r+1}$, i.e. the components $\hat{g}_{ij}$ of $\hat{g}$ in a coordinate domain of M are functions of type C^{r+1}. We denote by $\hat{\nabla}$ the Levi-Civita connection of $\hat{g}_{ij}$, and, evidently, $\hat{\nabla} \in C^r$.

We define $\tau(X) = \frac{1}{n+1}\, trace(V \mapsto \hat{\nabla}_V X)$ and we construct $\bar{\nabla}$ in the following way

$$\bar{\nabla}_X Y = \blacktriangledown_X Y + \tau(X) \cdot Y + \tau(Y) \cdot X. \tag{3}$$

It is easily seen that $\bar{\nabla}$ constructed in this way is an affine connection on M. The components of the object $\bar{\nabla}$ in the coordinate system (U, x) can be written in the form: $\bar{\Gamma}^h_{ij}(x) = \Pi^h_{ij}(x) + \tau_i(x) \cdot \delta^h_j + \tau_j(x) \cdot \delta^h_i$ where Π^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of the projective connection $\blacktriangledown$ and the affine connection $\bar{\nabla}$, respectively, and $\tau_i = \frac{1}{n+1}\, \partial G/\partial x^i$, $G = \ln \sqrt{|\det \|\hat{g}_{ij}\||}$. It is obvious that P_n is geodesically mapped onto $\bar{A}_n = (M, \bar{\nabla})$, and, evidently because $\bar{\Gamma}^h_{ij} \in C^r$, $\bar{A}_n \in C^r$.

Insofar as $\Pi^\alpha_{\alpha i}(x) = 0$, then $\bar{\Gamma}^\alpha_{\alpha i}(x) = \partial G/\partial x^i$, i.e. trace$(V \to \bar{\nabla}_V)X = \nabla_X G$. Hence follows that $\bar{A}_n$ has an equiaffine connection [26, p. 151]. Moreover, if $\nabla \in C^1$ then the Ricci tensor *Ric* is symmetric ([25, p. 35], [26, p. 151]). □

3. Geodesic Mappings from Equiaffine Manifolds onto (pseudo-) Riemannian Manifolds

Let manifold $A_n = (M, \nabla) \in C^0$ admit a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g}) \in C^1$, i.e. components $\bar{g}_{ij}(x) \in C^1(U)$. It is known [21], see [25, p. 145], that equations (1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik}. \tag{4}$$

If A_n is an equiaffine manifold then ψ have the following form

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1} \ln \sqrt{|\det \bar{g}|} - \rho, \quad \partial_i \rho = \frac{1}{n+1}\, \Gamma^\alpha_{\alpha i}, \quad \partial_i = \partial/\partial x^i,$$

and Mikeš and Berezovski [32], see [25, p. 150], proved that the Levi-Civita equations (1) and (4) are equivalent to

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k, \tag{5}$$

where

$$\text{(a)}\ a^{ij} = \mathrm{e}^{2\Psi}\, \bar{g}^{ij}; \quad \text{(b)}\ \lambda^i = -\,\mathrm{e}^{2\Psi}\, \bar{g}^{i\alpha} \psi_\alpha. \tag{6}$$

Here $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$. On the other hand:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \hat{g}_{ij}, \quad \Psi = \ln \sqrt{|\det \hat{g}|} - \rho, \quad \|\hat{g}_{ij}\| = \|a^{ij}\|^{-1}. \tag{7}$$

Using the equation $\Pi^h_{ij}(x) = \Gamma^h_{ij}(x)$ (see (37.4) in [5, p. 105]), where Π is a projective connection and Γ is *normal* affine connection (it is also equi-affine), we after substitution $\Gamma^h_{ij}(x) \mapsto \Pi^h_{ij}(x)$ into (5) have equation (2.3) in [4], immediately.

Furthermore, we assume that $A_n = (M, \nabla) \in C^1$ and $\bar{V}_n = (M, \bar{g}) \in C^2$. In this case, the integrability conditions of the equations (5) from the Ricci identity $\nabla_l \nabla_k a^{ij} - \nabla_k \nabla_l a^{ij} = -a^{i\alpha} R^j_{\alpha kl} - a^{j\alpha} R^i_{\alpha kl}$ have the following form

$$-a^{i\alpha} R^j_{\alpha kl} - a^{j\alpha} R^i_{\alpha kl} = \delta^i_k \nabla_l \lambda^j + \delta^j_k \nabla_l \lambda^i - \delta^i_l \nabla_k \lambda^j - \delta^j_l \nabla_k \lambda^i, \tag{8}$$

where R^h_{ijk} are components of the curvature (Riemannian) tensor R on A_n, and after contraction of the indices i and k we get [21]

$$n \nabla_l \lambda^j = \mu\, \delta^j_l - a^{j\alpha} R_{\alpha l} - a^{\alpha\beta} R^j_{\alpha\beta l}\,, \tag{9}$$

where $\mu = \nabla_\alpha \lambda^\alpha$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor *Ric* on A_n.

4. Main Theorems

Let $V_n = (M, g) \in C^r$ be the (pseudo-) Riemannian manifold. If $r \geq 1$ then its natural affine connection $\nabla \in C^{r-1}$ (i.e. the Levi-Civita connection) and projective connection $\blacktriangledown \in C^{r-1}$; hence $A_n = (M, \nabla)$ and $P_n = (M, \blacktriangledown)$ be manifolds with affine and projective connection, respectively. The following theorems are true.

Theorem 4.1. *If $P_n \in C^{r-1}$ $(r > 2)$ admits geodesic mappings onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

Theorem 4.2. *If $A_n \in C^{r-1}$ $(r > 2)$ admits geodesic mappings onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

Based on the previous comments (at the end of the second section), it will be sufficient to prove the validity of the second Theorem. Moreover, the manifold A_n can be an equiaffine manifold.

The proof of the Theorem 4.2 follows from the following lemmas.

Lemma 4.3 ([13]). *Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If $\partial_i \lambda^h - \varrho\,\delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.*

Proof. The condition $\partial_i \lambda^h - \varrho\,\delta^h_i \in C^1$ can be written in the following form

$$\partial_i \lambda^h - \varrho \delta^h_i = f^h_i(x), \tag{10}$$

where $f^h_i(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $h \neq i$ we integrate (10) with respect to dx^i:

$$\lambda^h = \Lambda^h + \int_{x^i_o}^{x^i} f^h_i(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt,$$

where Λ^h is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^h and the above integrals (see [17, p. 300]), also the derivatives $\partial_h \Lambda^h$ exist; in this proof we don't use Einstein's summation convention. Then we can write (10) for $h = i$:

$$\varrho = -f^h_h + \partial_h \Lambda^h + \int_{x^i_o}^{x^i} \partial_h f^h_i(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt. \tag{11}$$

Because the derivative with respect to x^i of the right-hand side of (11) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i \varrho = \partial_h f^h_i - \partial_i f^h_h$, therefore $\varrho \in C^1$ and from (10) follows $\lambda^h \in C^2$. $\square$

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i \lambda^h - \varrho \delta^h_i \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 4.4. *If $A_n \in C^2$ admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^3$.*

Proof. In this case Mikeš's and Berezovsky's equations (5) and (9) hold. According to the assumptions, $\Gamma^h_{ij} \in C^2$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda^i \in C^1$ and $R^h_{ijk}, R_{ij} \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (9) in the form (10), where

$$\varrho = \mu/n \quad \text{and} \quad f^h_i = (-\lambda^\alpha \Gamma^h_{\alpha i} + a^{j\alpha} R_{\alpha l} - a^{\alpha\beta} R^j_{\alpha\beta l})/n \in C^1.$$

From Lemma 4.3 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda^i \in C^2$. Differentiating (5) twice we convince ourselves that $a^{ij} \in C^3$. From this and formula (7) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. $\square$

Further we notice that for geodesic mappings from $A_n \in C^2$ onto $\bar{V}_n \in C^3$ holds the third set of Mikeš's and Berezovsky's equations [21]:

$$(n-1)\nabla_k \mu = -2(n+1)\lambda^\alpha R_{\alpha k} + a^{\alpha\beta}(R_{\alpha\beta,k} - 2R_{\alpha k,\beta}). \tag{12}$$

If $A_n \in C^{r-1}$ and $\bar{V}_n \in C^2$, then by Lemma 4.4, $\bar{V}_n \in C^3$ and (12) hold. Because Mikeš's and Berezovsky's system (5), (9) and (12) is closed, we can differentiate equations (5) r times. So we convince ourselves that $a^{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{V}_n \in C^r)$. $\square$

References

[1] A. V. Aminova, Projective transformations of pseudo-Riemannian manifolds, J. Math. Sci., New York 113 (2003) 367–470.
[2] H. Chudá, J. Mikeš, Conformally geodesic mappings satisfying a certain initial condition. Arch. Math. (Brno) 47 (2011), no. 5, 389–394.
[3] M. S. Ćirić, M. Lj. Zlatanović, M. S. Stanković, Lj. S. Velimirović, On geodesic mappings of equidistant generalized Riemannian spaces, Appl. Math. Comput. 218 (2012) 6648–6655.
[4] M. Eastwood, V. Matveev, Metric connections in projective differential geometry, The IMA Volumes in Math. and its Appl. 144 (2008) 339–350.
[5] L. P. Eisenhart, Non-Riemannian Geometry. Princeton Univ. Press. 1926. Amer. Math. Soc. Colloquium Publications 8 (2000).
[6] L. E. Evtushik, Yu. G. Lumiste, N. M. Ostianu, A. P. Shirokov, Differential-geometric structures on manifolds. J. Sov. Math. 14 (1980) 1573–1719; transl. from Itogi Nauki Tekh., Ser. Probl. Geom. 9 (1979).
[7] S. Formella, J. Mikeš, Geodesic mappings of Einstein spaces. Szczecińske rocz. naukove, Ann. Sci. Stetinenses. 9 (1994) 31–40.
[8] G. Hall, Projective structure in space-times. Adv. in Lorentzian geometry, AMS/IP Stud. Adv. Math. 49 (2011) 71–79.
[9] G. Hall, Z. Wang, Projective structure in 4-dimensional manifolds with positive definite metrics. J. Geom. Phys. 62 (2012) 449–463.
[10] I. Hinterleiner, Geodesic mappings on compact Riemannian manifolds with conditions on sectional curvature. Publ. Inst. Math. (Beograd) (N.S.) 94(108) (2013) 125–130.
[11] I. Hinterleiner, J. Mikeš, On the equations of conformally-projective harmonic mappings. XXVI Workshop on Geometrical Methods in Physics, 141–148, AIP Conf. Proc., 956, Amer. Inst. Phys., Melville, NY, 2007.
[12] I. Hinterleiner, J. Mikeš, Fundamental equations of geodesic mappings and their generalizations, J. Math. Sci. (N. Y.) 174 (2011) 537–554; transl. from Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow 124 (2010) 7–34.
[13] I. Hinterleiner, J. Mikeš, Projective equivalence and spaces with equi-affine connection, J. Math. Sci. (N. Y.) 177 (2011) 546–550; transl. from Fundam. Prikl. Mat. 16 (2010) 47–54.
[14] I. Hinterleiner, J. Mikeš, Geodesic Mappings and Einstein Spaces, Basel: Birkhäuser, Trends in Mathematics, (2013) 331–335.
[15] I. Hinterleiner, J. Mikeš, Geodesic Mappings of (pseudo-) Riemannian manifolds preserve class of differentialbility, Miskolc Mathematical Notes, 14 (2013) 89-96.
[16] M. Jukl, L. Lakomá, The Decomposition of Tensor Spaces with Almost Complex Structure, Rend. del Circ. Mat. di Palermo, Serie II, Suppl. 72 (2004) 145–150.
[17] L. D. Kudrjavcev, Kurs matematicheskogo analiza. Moscow, Vyssh. skola, 1981.
[18] T. Levi-Civita, Sulle transformationi delle equazioni dinamiche, Ann. Mat. Milano, 24 (1886) 255–300.
[19] J. Mikeš, Geodesic mappings of Einstein spaces. Math. Notes 28 (1981) 922–923. Transl. from Mat. Zametki 28 (1980), no. 6, 935–938.
[20] J. Mikeš, Geodesic mappings of affine-connected and Riemannian spaces. J. Math. Sci., New York 78 (1996) 311–333.
[21] J. Mikeš, V. Berezovski, Geodesic mappings of affine-connected spaces onto Riemannian spaces, Colloq. Math. Soc. János Bolyai. 56 (1992) 491–494.
[22] J. Mikeš, H. Chudá, On geodesic mappings with certain initial conditions. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 26 (2010), no. 2, 337–341.
[23] J. Mikeš, I. Hinterleitner, On geodesic mappings of manifolds with affine connection, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 26 (2010) 343–347.
[24] J. Mikeš, V. A. Kiosak, A. Vanžurová, Geodesic mappings of manifolds with affine connection, Palacky University Press, 2008.
[25] J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic mappings and some generalizations, Palacky University Press, 2009.
[26] A. P. Norden, Spaces of Affine Connection, Nauka, Moscow, 1976.
[27] A. Z. Petrov, New methods in the general theory of relativity. M., Nauka, 1966.
[28] M. Prvanović, Foundations of geometry. (Osnovi geometrije), (Serbo-Croat) Univ. Novy Sad, Prirodno-Matem. Fakultet, OOUR Inst. za Matem. Beograd: "Gradevinska Knjiga". XII, 1980.
[29] M. Prvanovitch, Projective and conformal transformations in recurrent and Ricci-recurrent Riemannian spaces, Tensor (N.S.) 12 (1962) 219–226.
[30] Zh. Radulovich, J. Mikeš, M. L. Gavril'chenko, Geodesic mappings and deformations of Riemannian spaces, CID, Podgorica, 1997.
[31] P. A. Shirokov, Selected investigations on geometry, Kazan' Univ. press, 1966.
[32] N. S. Sinyukov, Geodesic mappings of Riemannian spaces, Nauka, Moscow, 1979.
[33] A.S. Solodovnikov, Spaces with common geodesics, Tr. Semin. Vektor. Tenzor. Anal. 11 (1961) 43–102.
[34] M. S. Stanković, S. M. Mincić, Lj. S. Velimirović, M. Lj. Zlatanović, On equitorsion geodesic mappings of general affine connection spaces, Rend. Semin. Mat. Univ. Padova 124 (2010) 77–90.
[35] T. Y. Thomas, Determination of affine and metric spaces by their differential invariants, Math. Ann. 101 (1929) 713–728.
[36] G. Vrancęanu, Leçons de geometri différentielle, vol. I, II, Ed. de l'Acad. de la Rep. Popul. Roumaine, Bucharest, 1957.
[37] H. Weyl, Zur Infinitesimalgeometrie Einordnung der projektiven und der konformen Auffassung, Göttinger Nachrichten (1921) 99–112.